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On Jessen's Inequality for Convex Functions, III

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1. INTRODUCTION

Let E be a nonempty set and L be a linear class of real valued functions $g: E \to R$ having the properties:

(L1) $f, g \in L \Rightarrow (af + bg) \in L$ for all $a, b \in R$;

(L2) $\mathbb{1} \in L$, that is if $f(t) \equiv 1$ $(t \in E)$, then $f \in L$.

We also consider *isotonic linear functionals* $A: L \to R$. That is, we suppose

(A1) $A(af + bg) = aA(f) + bA(g)$ for $f, g \in L$, $a, b \in R$;

(A2) $f \in L$, $f(t) \geqq 0$ on $E \Rightarrow A(f) \geqq 0$ (A is isotonic).

In this paper we shall give an inverse of Jessens's inequality for convex functions [1, 2]:

Jessen's Inequality. Let L satisfy properties L1, L2 on a nonempty set E, and suppose f is a continuous convex function on an interval $I \subset R$. If A is any isotonic linear functional with $A(\mathbb{1}) = 1$, then, for all $g \in L$ such that $f \circ g \in L$, we have $A(g) \in I$ and

$$f(A(g)) \leqq A(f \circ g). \tag{1}$$

Section 3 gives generalizations of the well-known Aczél's, Popoviciu's, and Bellman's inequalities for isotonic functionals. Section 4 contains some general results for convex functions. Using these and a result from [2] we obtain a generalization of a result from [9]. Similar results are also given in Section 5.

2. SOME REVERSE INEQUALITIES

Reverse Jessen's Inequality. Let E, L, f be defined as in Jessen's inequality, and suppose that $p \in L$ with $p \geqq 0$ on E and $0 < A(p) < u$

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$(u \in R)$, $(ua - A(pg))/(u - A(p)) \in I$ $(a \in I)$, $pg \in L$ and $p \cdot f \circ g \in L$, where A is any isotonic functional. Then

$$f\left(\frac{ua - A(pg)}{u - A(p)}\right) \geqq \frac{uf(a) - A(p \cdot f \circ g)}{u - A(p)}. \tag{2}$$

Proof. The following inequality is valid for convex function f (see, for example, [4]):

$$f\left(\frac{pa + qb}{p+q}\right) \geqq \frac{pf(a) + qf(b)}{p+q} \qquad \left(q < 0,\ p + q > 0,\ a, b, \frac{pa + qb}{p+q} \in I\right).$$

By substitutions

$$p = u, \qquad q = -A(p), \qquad b = A(pq)/A(p),$$

we obtain

$$f\left(\frac{ua - A(pg)}{u - A(p)}\right) \geqq \frac{uf(a) - A(p)\, f(A(pg)/A(p))}{u - A(p)}.$$

Now, using Jessen's inequality, i.e.,

$$f\left(\frac{A(pg)}{A(p)}\right) \leqq \frac{A(p \cdot f \circ g)}{A(p)},$$

we obtain (2).

A generalization of Jessen's inequality is given in [5] (and recently in [6]). Using this result we could, similarly, give a generalization of (2), which is left to the reader.

3. Aczél's, Popoviciu's, and Bellman's Inequalities

Aczél's Inequality for Isotonic Functionals. Let L satisfy conditions L1, L2, and A satisfy conditions A1, A2 on a base set E. If f^2, g^2, $fg \in L$, and if

$$g_0^2 > A(g^2) > 0 \qquad (\text{or } f_0^2 > A(f^2) > 0),$$

where g_0, f_0 are real numbers, then

$$(f_0 g_0 - A(fg))^2 \geqq (f_0^2 - A(f^2))(g_0^2 - A(g^2)). \tag{3}$$

Proof of this result is similar to the proof of the following result:

Popoviciu's Inequality for Isotonic Functionals. Let A and L be as in the above result. If $p > 1$, $p^{-1} + q^{-1} = 1$, f, $g \geqq 0$ on E, f^p, g^q, $fg \in L$, and f_0, g_0 are positive numbers such that

$$g_0^q > A(g^q) > 0 \qquad \text{and} \qquad f_0^p > A(f^p) > 0, \tag{4}$$

then

$$(f_0^p - A(f^p))^{1/p} (g_0^q - A(g^q))^{1/q} \leqq f_0 g_0 - A(fg). \tag{5}$$

In case $p < 1$ ($p \neq 0$, $p^{-1} + q^{-1} = 1$) the inequality (5) is reversed.

Proof. By substitutions

$$f(x) = x^p \ (p > 1),\ u \to g_0^q,\ a \to g_0^{-q/p} f_0,\ P \to g^q,\ g \to g^{-q/p} f$$

we obtain from (2)

$$(f_0 g_0 - A(fg))^p \geqq (f_0^p - A(f^p))(g_0^q - A(g^q))^{p-1} \tag{6}$$

if the first condition in (4) is satisfied. If the second condition is also satisfied, then using Hölder's inequality for isotonic functional [1] we have

$$A(fg) < A(f^p)^{1/p} A(g^q)^{1/q} < f_0 g_0,$$

i.e.,

$$f_0 g_0 - A(fg) > 0,$$

and from (6) we obtain (5).

Remark. From the special case of (5), i.e., from

$$(a_1^p - a_2^p)^{1/p} (b_1^q - b_2^q)^{1/q} \leqq a_1 b_1 - a_2 b_2,$$

by substitutions

$$a_1 \to f_0,\ a_2 \to A(f^p)^{1/p},\ b_1 \to g_0,\ b_2 \to A(g^q)^{1/q},$$

and by Hölder's inequality for isotonic functionals, we obtain

$$\begin{aligned}(f_0^p - A(f^p))^{1/p} &(g_0^q - A(g^q))^{1/q} \\ &\leqq f_0 g_0 - A(f^p)^{1/p} A(g^q)^{1/q} \leqq f_0 g_0 - A(fg)\end{aligned} \tag{7}$$

for $p > 1$, and the reverse inequality for $p < 1$ ($p \neq 0$).

This is a generalization of [7, inequality (25)].

Bellman's Inequality for Isotonic Functionals. Let A and L be as in the previous results. If $p>1$ and if $f, g \geqq 0$ on E with $f^p, g^p, (f+g)^p \in L$, and $f_0\, g_0$ are positive numbers such that

$$g_0^p > A(g^p) > 0 \qquad \text{and} \qquad f_0^p > A(f^p) > 0 \tag{8}$$

then

$$\begin{aligned}((f_0^p - A(f^p))^{1/p} + (g_0^p - A(g^p))^{1/p})^p \\ \leqq (f_0 + g_0)^p - A((f+g)^p).\end{aligned} \tag{9}$$

The inequality (9) is reversed is $p<1$ $(p \neq 0)$.

Proof. We shall give proof only for $p>1$. Using (8) and Minkowski's inequality we have

$$(f_0+g_0)^p > (A(f^p)^{1/p} + A(g^p)^{1/p})^p \geqq A((f+g)^p).$$

Now, we shall use discrete Minkowski's inequality for $n=2$, i.e.,

$$((a_1+b_1)^p + (a_2+b_2)^p)^{1/p} \leqq (a_1^p + a_2^p)^{1/p} + (b_1^p + b_2^p)^{1/p}.$$

By using the substitutions,

$$a_1^p \to f_0^p - A(f^p),\ b_1^p \to g_0^p - A(g^p),\ a_2^p \to A(f^p),\ b_2^p \to A(g^p)$$

and Minkowski's inequality for isotonic functionals we obtain

$$\begin{aligned}((f_0^p - A(f^p))^{1/p} + (g_0^p - A(g^p))^{1/p})^p &\leqq (f_0+g_0)^p - (A(f^p)^{1/p} + A(g^p)^{1/p})^p \\ &\leqq (f_0+g_0)^p - A((f+g)^p).\end{aligned}$$

Remark 2. The last inequality is an interpolation of (9), and it is a generalization of [7, Lemma 4].

4. Remark on General Linear Inequalities for Convex Functions with some Applications

4.1. Let $D \subset R$ and let $S(D)$ be one of the normed subspaces of the space of all real functions defined on D. Further let the set of all continuous convex functions on $[a, b]$ be denoted by $K[a, b]$, and

$$e_0(t) = 1, \qquad e_1(t) = t, \qquad w(t, c) = |t - c|.$$

The following result is given in [8].

Let us assume that the operator $A: C[a, b] \to S(D)$ is linear and continuous. Then for every function $t \mapsto f(t)$ the following implication

$$f \in K[a, b] \Rightarrow Af \geqq 0 \tag{10}$$

is valid if and only if the following three conditions all hold:

$$Ae_0 = 0, \tag{11}$$

$$Ae_1 = 0, \tag{12}$$

$$Aw(t, c) \geqq 0 \qquad \text{for every } c \in [a, b]. \tag{13}$$

Here, we shall note that the condition (12) is superfluous, since it is implied by (11) and (13). A similar remark holds for [8, Theorems 3–5]. Indeed we have

$$Aw(t, a) = Ae_1 - aAe_0 = Ae_1 \geqq 0$$
$$Aw(t, b) = bAe_0 - Ae_1 = -Ae_1 \geqq 0$$

where we used (11). Of course, it is possible only if (12) is valid, i.e., (11) and (13) imply (12). Therefore, we can formulate the above result without (12).

Using this result and [9, Lemma 4] we obtain the following result:

LEMMA. *The inequality*

$$\sum_{i=1}^{n} p_i f(x_i) \geqq 0 \tag{14}$$

holds for all n-tuples x and p and for every convex function f if and only if

$$\sum_{i=1}^{n} p_i = 0, \tag{15}$$

and for $k = 1, ..., n$,

$$\sum_{i=1}^{n} p_i \, |a_i - a_k| \geqq 0. \tag{16}$$

4.2 In the further text we shall consider positive functions for which $A(f) > 0$. Also, we shall suppose that there exists an interval I from R, where $A(f^r)$ $(r \in I)$ is well defined.

The following result is proved in [2].

The function $G(r) = A(f^r)$ is logarithmically convex (and so is convex) on I.

Therefore, if (15) and (16) are valid, then

$$1 \leqq A(f^{x_1})^{p_1} \cdots A(f^{x_n})^{p_n} \qquad (x_i \in I,\ i = 1, ..., n). \tag{17}$$

This is a generalization of [9, inequality (5)]. Similar generalizations of other results from [9] can be also given.

4.3. Now, we shall give a similar result.

First, note that Popoviciu's inequality for isotonic functionals can be stated in the form

$$(f_0 - A(f))^s (g_0 - A(g))^{1-s} \leqq f_0^s g_0^{1-s} - A(f^s g^{1-s}), \tag{18}$$

where $s \in (0, 1)$, and where $A(f) < f_0$ and $A(g) < g_0$.

We can consider the function

$$P(r) = f_0^r - A(f^r),$$

positive on an interval J. We have

$$\begin{aligned} P(\lambda r + (1-\lambda) s) &= (f_0^r)^\lambda (f_0^s)^{1-\lambda} - A((f^r)^\lambda (f^s)^{1-\lambda}) \\ &\geqq (f_0^r - A(f^r))^\lambda (f_0^s - A(f^s))^{1-\lambda} = P(r)^\lambda P(s)^{1-\lambda}. \end{aligned}$$

Hence, this function is logarithmically concave on J.

As a consequence of our lemma, we have the following result:

Let x and p be two real n-tuples such that $x_i \in J$ $(i = 1, ..., n)$ and (15) and (16) are valid. Then

$$1 \geqq \prod_{i=1}^{n} (f_0^{x_i} - A(f^{x_i}))^{p_i}. \tag{19}$$

4.4 The above property of function $G(r)$ is used in the proof of the following result. Let [10]

$$g(x) = \prod_{j=1}^{n} A\left(f_j^{q_j x} \prod_{k=1}^{n} f_k^{r-x}\right)^{1/q_j},$$

where $q_i > 0$ $(i = 1, ..., n)$ with $\sum_{k=1}^{n} 1/q_k = 1$, $r \in R$, and all expressions $A(\cdot)$ are well defined.

If $|x| \leqq |y|$ $(xy > 0)$, then

$$g(x) \leqq g(y). \tag{20}$$

The function g is also logarithmically convex (and so is convex). The reverse results are valid if we consider the function

$$g(x) = \prod_{j=1}^{n} \left(f_{jo}^{q_j x} \prod_{k=1}^{n} f_{ko}^{r-x} - A\left(f_j^{q_j x} \prod_{k=1}^{n} f_k^{r-x}\right)\right)^{1/q_j},$$

i.e., we have reverse inequality in (20) and g is logarithmically concave function in an interval, where it is well defined.

These results are generalizations of Hölder's and Popoviciu's inequalities.

5. Some Further Remarks about Hölder's Inequality

5.1. The condition $p^{-1}+q^{-1}=1$ for real numbers p, q, in Hölder's inequality can be replaced by

$$p^{-1}+q^{-1}=r^{-1} \qquad (p, q, r \in R\backslash\{0\}. \tag{21}$$

The corresponding results follow from Hölder's inequality itself by substitutions

$$f \to f^r, g \to g^r, p \to p/r, q \to q/r. \tag{22}$$

So, the following result is valid.

If wf^p, wg^q, $wf^rg^r \in L$, then inequality

$$A(wf^rg^r)^{1/r} \leqq A(wf^p)^{1/p} A(wg^q)^{1/q}, \tag{23}$$

holds if either

$$\text{(i) } p, q, r>0; \qquad \text{or (ii) } p, -q, -r>0; \qquad \text{or (iii) } -p, q, -r>0,$$

is valid. If either

$$\text{(iv) } p, q, r<0; \qquad \text{or (v) } -p, q, r>0; \qquad \text{or (vi) } p, -q, r>0$$

is valid, then the inequality in (23) is reversed.

Further, if $0<m \leqq f(x)^r g(x)^{-qr/p} \leqq M$ for $x \in E$, then from [1, Theorem 10] it follows:

$$A(wf^rg^r)^{1/r} \geqq |p|^{1/p} |q|^{1/q} |r|^{-1/r} \frac{(M-m)^{1/p} |mM^{p/r}-Mm^{p/r}|^{1/q}}{|M^{p/r}-m^{p/r}|^{1/r}} \times A(wf^p)^{1/p} A(wg^q)^{1/q}, \tag{24}$$

in cases (i)–(iii). The inequality is reversed in cases (iv)–(vi).

The above results are generalizations of results from [11, 12]. Similarly, [1, Theorem 9] gives

$$(M-m) A(wf^p)+(mM^{p/r}-Mm^{p/r}) A(wg^q) \leqq (M^{p/r}-m^{p/r}) A(wf^rg^r) \tag{25}$$

in cases (i), (ii), (iv), (v) and the reverse inequality in the cases (iii), (vi).

The same substitution can be used in some other generalizations of Hölder's inequality. For example, one can give similar generalization of Beckenbach's generalization of Hölder's inequality, i.e., of the generalization given in [13, Corollary 3].

5.2. Now, we shall give further generalizations of two results from [2, 3.1 and 3.2].

Denote by $S^{[r]}(g; A) := A(g^r)^{1/r}$ $(r \neq 0)$. If $A(\mathbb{1}) = 1$ we have the potential mean for isotonic functionals defined in [1, 2].

Liapunov's inequality for isotonic functionals [2, 3.1] becomes

$$S^{[s]}(g; A) \leqq S^{[r]}(g; A)^{(r/s)((r-s)/(r-t))} S^{[t]}(g; A)^{(t/s)((s-t)/(r-t))} \qquad (0 < t < s < r). \tag{26}$$

Hence, by the arithmetic–geometric mean inequality

$$S^{[s]}(g; A) \leqq \frac{r}{s}\frac{r-s}{r-t} S^{[r]}(g; A) + \frac{t}{s}\frac{s-t}{r-t} S^{[t]}(g; A)$$

or for $S^{[r]}(g; A) \geqq S^{[s]}(g; A)$

$$\frac{S^{[r]}(g; A) - S^{[t]}(g; A)}{S^{[r]}(g; A) - S^{[s]}(g; A)} \leqq \frac{s(r-t)}{t(r-s)} \qquad (0 < t < s < r),$$

and reverse inequality for $S^{[r]}(g; A) \leqq S^{[s]}(g; A)$.

As in [2, 3.2] we can prove that the function $f(s) = S^{[1/s]}(g; A)$ is logarithmically convex (hence convex) for $s > 0$.

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